# A Finite-Element Approach for Nonlinear Panel Flutter

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**A finite-element approach has been developed for determining nonlinear flutter characteristics of two-dimensional panels, based on aerodynamic forces from quasi-steady aerodynamic theory. The stiffness equations of motion, iterative solution procedure, and convergence characteristics are presented. Comparisons are made with linear flutter and large amplitude vibration results and demonstrate that good accuracy is obtained. Nonlinear flutter results are presented; effects of aerodynamic damping, boundary support conditions, and initial in-plane forces are included.**

## Introduction

PANEL flutter is the self-excited oscillation of the external skin of a flight vehicle when exposed to an airflow along its surface. The classic approach, using linear structural theory, indicates that there is a critical (or flutter) dynamic pressure above which the panel motion becomes unstable and grows exponentially with time. Since the linear theory does not account for structural nonlinearities, it can determine only the flutter boundary, and can give no information about the flutter oscillation itself. A great quantity of literature exists on linear panel flutter (e.g., Refs. 1 and 2, plus others too numerous to mention).

For large deflections, the nonlinear effects, mainly due to midplane stretching forces, restrain the panel motion to bounded limit cycle oscillations with increasing amplitude as dynamic pressure increases. Therefore, for realistic assessments and understanding of panel flutter, the nonlinear structural theory should be used. An excellent survey on both linear and nonlinear panel flutter through 1970 is given by Dowell.[3]

In order to investigate large amplitude panel flutter, a number of approaches can be used. A modal approach with direct numerical integration has been used by Dowell.[4,5] The major disadvantage in using this approach is its long computing time. The harmonic balance method can be used to determine limit cycles; see for example, Eastep and McIntosh[6] and Kuo et al.[7] This approach, however, is quite complicated in mathematic manipulations. Morino[7,8] also used the perturbation method to obtain neighboring solutions to the linear problem.

The finite-element method has been used successfully in investigating linear panel flutter.[9-17] Because of its versatile applicability, effects of aerodynamic damping, complex panel configuration (e.g. delta planform[11] and rhombic planform[13]), flow angularity, midplane forces, and anisotropic material properties can be conveniently included. Recently, the finite-element method has been applied successfully in large amplitude vibrations of beam and plate structures.[18-21] In this study, the finite-element formulation given in Ref. 18 is extended to treat the limit cycle oscillations of two-dimensional panels. This note includes a brief discussion of the theoretical formulation and solution procedure. Effects of aerodynamic damping, initial in-plane loading, and boundary support condition are included. Comparison of nonlinear vibration and linear flutter results with analytical solutions demonstrate that excellent accuracy is obtained.

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## Formulation and Its Solution

Consider a two-dimensional flat plate of length $a$, thickness $h$, and mass per unit area $m$, with air flowing above at supersonic Mach number $M$ in the positive $x$ direction. The differential equation of motion is

$$D\frac{\partial^4 w}{\partial x^4} - (N_x + N_{xo})\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = p \tag{1}$$

where

$$N_x = \frac{Eh}{2a}\int_0^a \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{2}$$

is the membrane force induced by large deflections, and $N_{xo}$ is the initial in-plane loading (tension positive), and $D = Eh^3/12(1-\nu^2)$ is the bending rigidity. For sufficiently high supersonic speeds ($M > 1.6$), the aerodynamic pressure can be described by the two-dimensional aerodynamic theory:

$$p(x,y,t) = -\frac{2q}{\beta}\left[\frac{\partial w}{\partial x} + \frac{1}{V}\frac{M^2-2}{M^2-1}\frac{\partial w}{\partial t}\right] \tag{3}$$

where $V$ is the air flow velocity, $q = \rho V^2/2$ is the dynamic pressure, $\rho$ is the air density, and $\beta = (M^2-1)^{1/2}$. In the finite-element method, the stiffness equations of motion for a plate element for Eq. (1) under the influence of elastic, initial in-plane, large deflection and inertia forces[19] with the inclusion

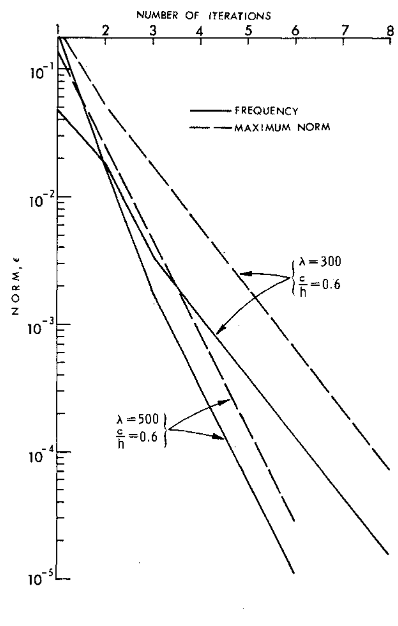


**Fig. 1 Convergence characteristics.**

of aerodynamic forces[9,11] may be written as:

$$([k]+[n]+[g])\{u_e\}+[m]\{\ddot{u}_e\}+[a]\{u_e\}=\{F\} \tag{4}$$

The stiffness $[k]$, stability stiffness $[n]$, and mass $[m]$ matrices have been well developed for almost every plate finite-element available. The geometrical stiffness matrix $[g]$ was formulated for a beam[18] and a rectangular plate element.[19] The aerodynamic matrix $[a]$ was formulated by Olson for a two-dimensional plate,[9] and later for two rectangular plate and a triangular plate elements.[11]

Assuming the displacements are exponential functions of time

$$\{u_e\}=\{u_e\}_o e^{\Omega t} \tag{5}$$

where, in general, $\Omega$ is a complex number, $\Omega=\alpha+i\omega$. The equation of motion, Eq. (4), for a two-dimensional plate element takes the form that

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \left\{ \frac{2D}{\ell} \begin{bmatrix} 6/\ell^2 & & & \text{sym} \\ 3/\ell & 2 & & \\ -6/\ell^2 & -3/\ell & 6/\ell^2 & \\ 3/\ell & 1 & -3/\ell & 2 \end{bmatrix} \right.$$

$$+\left(N_{xo}+\frac{N_x}{2}\right)\begin{bmatrix} 6/5\ell & & & \text{sym} \\ 1/10 & 2\ell/15 & & \\ -6/5\ell & -1/10 & 6/5\ell & \\ 1/10 & -\ell/30 & -1/10 & 2\ell/15 \end{bmatrix}$$

$$+\left(m\Omega^2+\frac{2q}{\beta V}\frac{M^2-2}{M^2-1}\Omega\right)\frac{\ell}{420}$$

$$\times\begin{bmatrix} 156 & & & \text{sym} \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell & -22\ell & 4\ell^2 \end{bmatrix}$$

$$\left. +\frac{2q}{\beta}\begin{bmatrix} -1/2 & \ell/10 & 1/2 & -\ell/10 \\ -\ell/10 & 0 & \ell/10 & -\ell^2/60 \\ -1/2 & -\ell/10 & 1/2 & \ell/10 \\ \ell/10 & \ell^2/60 & -\ell/10 & 0 \end{bmatrix}\right\} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{6}$$

where

$$N_x=\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}^T \frac{Eh}{2\ell} \times \begin{bmatrix} 6/5\ell & & & \text{sym} \\ 1/10 & 2\ell/15 & & \\ -6/5\ell & -1/10 & 6/5\ell & \\ 1/10 & -\ell/30 & -1/10 & 2\ell/15 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{7}$$

and $\ell$ is the length of the element.

Assembling the finite elements, applying the kinematic boundary conditions, and dividing by $(D/a^3)$, Eq. (6) leads to a nondimensional eigenvalue problem of the form:

$$([K]+[N]+[G]+\lambda[A]-\kappa[M])\{u\}=0 \tag{8}$$

where $\lambda=2q\,a^3/\beta D$ and $\kappa=-m\,a^4\Omega^2/D-\lambda a\,(M^2-2)\,\Omega/\beta^2 V$ are the nondimensional dynamic pressure parameter and eigenvalues, respectively. The eigenvalues can be put into more convenient form as

$$\kappa=-(\Omega/\omega_o)^2-g_A(\Omega/\omega_o) \tag{9}$$

where $g_A=(M^2-2)\ \rho V/\beta^3 m\omega_o$ is the nondimensional aerodynamic damping parameter and $\omega_o=(D/ma^4)^{1/2}$ is a convenient frequency scale. For typical panels, $g_A$ ranges from 0 to 50 approximately, as given in Fig. 2 of Ref. 2.

In determining the eigenvalues $\kappa$ in Eq. (8) for a given dynamic pressure $\lambda$, the iterative procedure and equivalent linearization technique discussed in detail in Ref. 20 was employed. The solution procedure is illustrated briefly as follows. For a given $\lambda$, first the linear flutter problem is solved

$$\kappa[M]\{\phi\}_0=([K]+[N]+\lambda[A])\{\phi\}_0 \tag{10}$$

where $\{\phi\}_0$ represents the linear mode shape normalized by its maximum components. The first approximate displacement is $\{u\}_1=c\,\text{real}(\{\phi\}_0 e^{(\alpha+i\omega)t})$ where $c$ is a given amplitude of panel oscillations, and $\alpha$ and $\omega$ are the panel response parameters related to $\kappa$ and $g_A$ by Eq. (14). An equivalent geometrical stiffness matrix $[G]_{\text{eq}}$ now can be obtained using $\{u\}_1$, and Eq. (8) is approximated by a linearized eigenvalue equation of the form

$$\kappa[M]\{\phi\}_1=([K]+[N]+[G]_{\text{eq}}+\lambda[A])\{\phi\}_1 \tag{11}$$

where $\kappa$ is the eigenvalue associated with amplitude $c$, and $\{\phi\}_1$ is the corresponding mode shape. The iterative process can be repeated until a convergence criterion is satisfied. All three displacement convergence criteria, the modified absolute norm, the modified Euclidean norm, and the maximum norm, proposed by Bergan and Clough,[22] were used in the present study. In addition, a frequency norm also is introduced and is defined as $\|\epsilon\|_f=|\Delta\kappa_n/\kappa_n|$ where $\Delta\kappa_n$ is the change in eigenvalue during iteration cycle $n$. A typical plot of the maximum and frequency norm vs number of iterations for a simply supported panel is shown in Fig. 1. The modified absolute norm and modified Euclidean norm fall in between the maximum and frequency norms, and therefore are not plotted on the figure. In the examples presented in the following section, convergence is considered achieved whenever any one of the norms reaches a value of $10^{-3}$.

Equation (8) indicates that when $\lambda=0$ the problem degenerates into large amplitude vibrations of in-vacuo panels. The matrices $[K]$, $[N]$, $[G]$, and $[M]$ are all symmetric, and the eigenvalues are real and positive. As $\lambda$ is increased from zero, two of these eigenvalues usually will approach each other and coalesce to $\kappa_{cr}$ at $\lambda=\lambda_{cr}$, and become complex conjugate pairs

$$\kappa=\kappa_R\pm i\kappa_I \tag{12}$$

for $\lambda>\lambda_{cr}$. Here $\lambda_{cr}$ is considered to be the lowest value of $\lambda$ for which coalescence occurs among all limit cycle amplitudes, and usually corresponds to $c=0$. A typical plot of $\kappa$ vs $\lambda$ is shown in Fig. 2. In the absence of aerodynamic damping ($g_A=0$), the flutter boundary simply corresponds to $\lambda_{cr}$. When $\lambda$ is below $\lambda_{cr}$, any disturbance to the panel decays and $(c/h)\to 0$.

For $\lambda>\lambda_{cr}$, a limit cycle oscillation exists which increases in amplitude as $\lambda$ increases. This can be seen more clearly by noting that the eigenvalue with a negative imaginary part

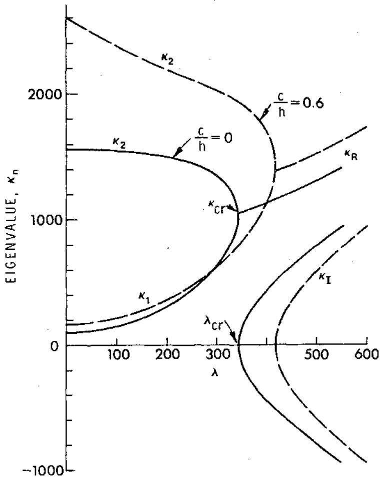


**Fig. 2 Variation of eigenvalues with dynamic pressure for simply supported panel ($N_{xo}=0$).**

leads to an instability (see Ref. 13) and relating the complex eigenvalues to the panel response parameters $\alpha$ and $\omega$ as follows: Rewrite Eqs. (9) and (12) as

$$(\Omega/\omega_o)^2+g_A(\Omega/\omega_o)+(\kappa_R-i\kappa_I)=0 \tag{13}$$

which can be solved for $\Omega$ to give

$$\Omega/\omega_o=(\alpha+i\omega)/\omega_o=(-g_A/2+\psi)+i(\kappa_I/2\psi) \tag{14}$$

where

$$\psi=\pm\left(\frac{1}{2}\left\{\sqrt{\left[\left(\frac{g_A}{2}\right)^2-\kappa_R\right]^2+\kappa_I^2}+\left[\left(\frac{g_A}{2}\right)^2-\kappa_R\right]\right\}\right)^{1/2} \tag{15}$$

The complete panel behavior is characterized by plotting the variation of $(\alpha+i\omega)$ with increasing dynamic pressure $\lambda$. Amplitude increases when $\alpha$ becomes positive. A typical plot is shown in Fig. 3.

## Results and Discussion

### A. Convergence Study

Numerical results for the first two eigenvalues at $\lambda=0$, and for the coalescence for a simply supported panel and a clamped panel are shown in Table 1. It is seen that an excellent approximation to the exact results[23] is obtained with only eight elements.

The influence of large deflections on in-vacuo frequencies for a simply supported panel is given in Table 2. Analytical solutions using three different approaches from Ref. 24 also are given. Comparison of the results show that the eight-element approximation gives very good accuracy. Therefore,

**Table 1 In-vacuo eigenvalues and coalescence results for simply supported and clamped panels**

| Number of elements | In-vacuo $\kappa_1$ | In-vacuo $\kappa_2$ | Coalescence $\lambda_{cr}$ | Coalescence $\kappa_{cr}$ |
|---|---|---|---|---|
| Simply supported panel | | | | |
| 2 | 98.1795 | 1920.00 | 398.536 | 1206.32 |
| 4 | 97.4597 | 1570.87 | 342.347 | 1043.47 |
| 8 | 97.4123 | 1559.35 | 343.280 | 1051.22 |
| Exact[23] | 97.4091 | 1558.55 | 343.3564 | 1051.797 |
| Clamped panel | | | | |
| 2 | 516.923 | 6720.00 | 922.388 | 3618.46 |
| 4 | 501.894 | 3874.23 | 636.437 | 2721.38 |
| 8 | 500.648 | 3808.34 | 636.586 | 2740.16 |
| Exact[23] | 500.564 | 3803.54 | 636.5691 | 2741.360 |

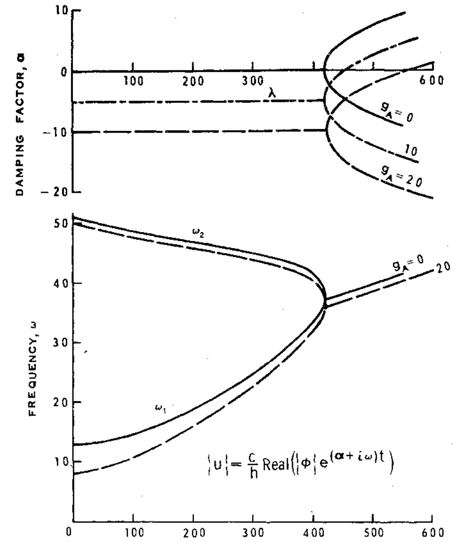


**Fig. 3 Typical plots of panel behavior and effect of aerodynamic damping (simply supported panel, $c/h=0.6$ and $N_{xo}=0$).**

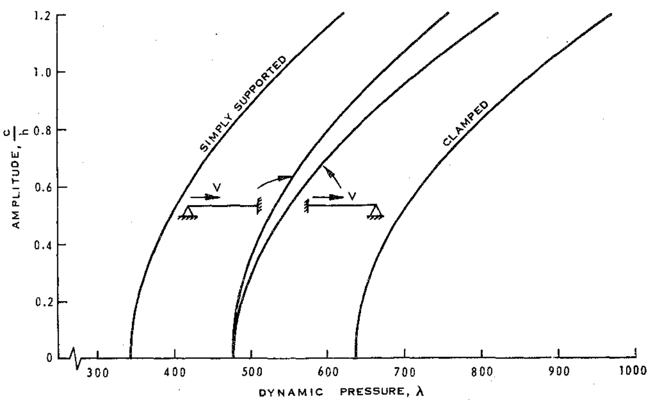


**Fig. 4 Limit cycle amplitude vs dynamic pressure for panels with various support conditions ($N_{xo}=g_A=0$).**

eight elements were used in modeling the panels in all the flutter results presented.

### B. Simply Supported Panel and Effect of Aerodynamic Damping

Plots of the eigenvalues vs dynamic pressure for a simply supported panel at two different panel amplitudes, $c/h=0.0$ (linear theory) and 0.6, are shown in Fig. 2. The complete panel behavior is characterized by plotting the $(\alpha+i\omega)$ variation with increasing dynamic pressure $\lambda$, using Eq. (14) and Fig. 2, as shown in Fig. 3. For the case of negligible aerodynamic damping, $g_A\rightarrow 0$, instability does not set in until after the two undamped natural frequencies have merged. If some damping is present, the instability sets in at a somewhat higher value, as indicated in Fig. 3. This occurs when $\alpha=0$ in Eq. (14). By routine algebraic manipulation, this instability occurs at the value of $\lambda$ when $g_A \doteq \kappa_I/(\kappa_R)^{1/2}$ and the corresponding limit cycle frequency is $\omega/\omega_0=(\kappa_R)^{1/2}$. However, as discussed earlier, this instability is not catastrophic. The panel response does not grow indefinitely, but rather a limit cycle oscillation is developed with increasing amplitude as $\lambda$ increases.

### C. Boundary Support Effect

In Fig. 4, the panel amplitude of the limit cycle oscillation is given as a function of $\lambda$ for various panel edge restraints. The most interesting result is that the limit cycle motions are different for hinged-clamped and clamped-hinged panels. This occurs because the aerodynamic matrices are different for the two support conditions, which leads to different

**Table 2 Effect of amplitude ratio on in-vacuo frequency ratios $(\omega/\omega_o)_n$ for simply supported panel**

| Amplitude $(c/h)$ | Mode $n$ | Number of elements 4 | 8 | 12 | Theory[24] Assumed space mode | Assumed time mode | Galerkin |
|---|---|---|---|---|---|---|---|
| 0.0 | 1 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 2 | 1.004 | 1.000 | 1.000 | 1.000 | ... | ... |
| 0.2 | 1 | 1.038 | 1.039 | 1.040 | 1.056 | 1.032 | 1.048 |
| | 2 | 1.030 | 1.038 | 1.039 | 1.056 | ... | ... |
| 0.4 | 1 | 1.141 | 1.147 | 1.148 | 1.206 | 1.124 | 1.181 |
| | 2 | 1.106 | 1.141 | 1.146 | 1.206 | ... | ... |
| 0.6 | 1 | 1.292 | 1.304 | 1.306 | 1.411 | 1.262 | 1.375 |
| | 2 | 1.221 | 1.292 | 1.301 | 1.411 | ... | ... |
| 0.8 | 1 | 1.471 | 1.489 | 1.492 | 1.647 | 1.434 | 1.607 |
| | 2 | 1.367 | 1.471 | 1.484 | 1.647 | ... | ... |
| 1.0 | 1 | 1.667 | 1.690 | 1.693 | 1.902 | 1.627 | 1.863 |
| | 2 | 1.534 | 1.667 | 1.685 | 1.902 | ... | ... |
| 1.2 | 1 | 1.869 | 1.902 | 1.906 | 2.167 | 1.837 | 2.136 |
| | 2 | 1.716 | 1.870 | 1.895 | 2.167 | ... | ... |

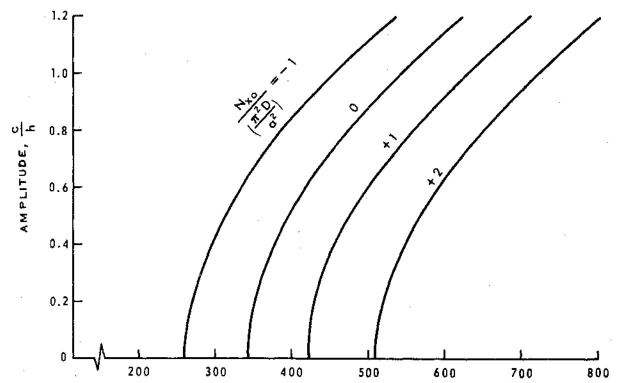


**Fig. 5 Limit cycle amplitude vs dynamic pressure for simply supported panel under different in-plane forces ($g_A = 0$).**

deflection shapes for the panels, as well as different geometrical stiffness matrices.

### D. Effect of In-Plane Loading

Panel amplitude vs λ for several applied in-plane forces acting on a simply supported panel is shown in Fig. 5. The classical Euler buckling load for simply supported panels is $N_{cr} = -\pi^2 D/a^2$. The total membrane force is composed of the applied in-plane load $N_{xo}$ and the membrane force $N_x$ induced by large deflections of the panel. Figure 5 shows that the applied compressive in-plane force reduces the critical dynamic pressure. However, as the dynamic pressure is increased, the panel amplitude increases, which induces tensile in-plane forces that counteract the applied compressive forces. This process continues until a flutter dynamic pressure is reached which corresponds to a given limit cycle amplitude.

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